Lower bound on spatial asymptotic of parabolic Anderson model

Fei Pu (Beijing Normal University)

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Parabolic Anderson model

$$\begin{cases} \partial_t u(t\,,x) = \frac{1}{2}\partial_x^2 u(t\,,x) + u(t\,,x)\,\xi(t\,,x), & t>0, x\in\mathbb{R}, \\ u(0,\cdot)\equiv 1, \end{cases} \tag{PAM}$$

where ξ denotes space-time white noise.

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- Integral equation

$$u(t\,,x) = 1 + \int_0^t \int_{\mathbb{R}} \boldsymbol{p}_{t-s}(x-y) u(s\,,y)\,\xi(\mathrm{d}s\,\mathrm{d}y),$$

with $\boldsymbol{p}_t(x) = (2\pi t)^{-1/2} \mathrm{e}^{-x^2/(2t)}$.

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  with $\boldsymbol{p}_t(x) = (2\pi t)^{-1/2}\mathrm{e}^{-x^2/(2t)}$.
- Conus-Joseph-Khoshnevisan-2013: there exist positive constants c_1, c_2 such that a.s.

$$c_1 \le \liminf_{N\to\infty} \frac{\max_{0\le x\le N}\log u(t\,,x)}{(\log N)^{2/3}} \le \limsup_{N\to\infty} \frac{\max_{0\le x\le N}\log u(t\,,x)}{(\log N)^{2/3}} \le C_2.$$

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- Fix $\beta > 0$. Let $\{U^{(\beta)}(t\,,x) : t \geq 0, x \in \mathbb{R}\}$ solve

$$U^{(\beta)}(t\,,x) = 1 + \int_0^t \int_{x-\sqrt{\beta}t}^{x+\sqrt{\beta}t} \boldsymbol{p}_{t-s}(x-y) U^{(\beta)}(s\,,y)\, \xi(\mathrm{d}s\, \mathrm{d}y).$$

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- Let $U^{(\beta,n)}(t\,,x)$ be the n-th Picard-iteration approximation to $U^{(\beta)}(t\,,x)$:

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- $u(t\,,x) \asymp U^{(\beta,[\log\beta])}(t\,,x)$ when β is large. Fix $t, \beta > 0$. $U^{(\beta,[\log\beta])}(t\,,x)$ and $U^{(\beta,[\log\beta])}(t\,,y)$ are independent if $|x-y| \geq \sqrt{\beta t}\log\beta$.

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- Chen-2016

$$\lim_{N\to\infty} \frac{\max_{0\leq x\leq N} \log u(t\,,x)}{(\log N)^{2/3}} = \frac{3}{4}\left(\frac{2t}{3}\right)^{1/3}, \quad \text{a.s.}$$

One of the key ingredients is

$$\lim_{m\to\infty} m^{-3} \log \mathrm{E}[u(t,0)^m] = \frac{t}{24}.$$

Parabolic Anderson Model with narrow wedge initial data

$$\begin{cases} \partial_t u(t\,,x) = \frac{1}{2}\partial_x^2 u(t\,,x) + u(t\,,x)\,\xi(t\,,x), & t>0, x\in\mathbb{R}, \\ u(0,\cdot) = \delta_0. \end{cases}$$

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Let $U(t,x) = \frac{u(t,x)}{\boldsymbol{p}_t(x)}$. Then

$$U(t,x) = 1 + \int_0^t \int_{\mathbb{R}} \boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}x\right) U(s,y)\,\xi(\mathrm{d}s\,\mathrm{d}y).$$

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- Huang-Lê-2019 proved

$$\limsup_{N\to\infty} \frac{\max_{0\le x\le N} \log U(t\,,x)}{(\log N)^{2/3}} \le \frac{3}{4}\left(\frac{2t}{3}\right)^{1/3} \quad \text{a.s.}$$

and they conjectured that

$$\lim_{N\to\infty} \frac{\max_{0\le x\le N} \log U(t\,,x)}{(\log N)^{2/3}} = \frac{3}{4}\left(\frac{2t}{3}\right)^{1/3} \quad \text{a.s.}$$

Theorem 1. (P.-2023+)

$$\liminf_{N\to\infty} \frac{\max_{0\le x\le N} \log U(t,x)}{(\log N)^{2/3}} \ge \frac{1}{4}\left(\frac{t}{2}\right)^{1/3} \quad \text{a.s.}$$

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- Corwin-Ghosal-2020 give lower and upper bounds on the moments of $U(t\,,0)$, which leads to

$$\lim_{m\to\infty} m^{-3} \log \mathrm{E}[U(t,0)^m] = \frac{t}{24}.$$

- Ganguly-Hegde-2022 estimate the upper tail probability, which leads to

$$\lim_{\theta\to+\infty} \frac{\log \mathrm{P}\left(\log U(t\,,0) \ge \theta\right)}{\theta^{3/2}} = -\frac{4}{3}\sqrt{\frac{2}{t}}.$$

Association

- Chen-Khoshnevisan-Nualart-P.-2023: $\{U(t\,,x) : t > 0, x \in \mathbb{R}\}$ is associated, i.e.,

$$\mathrm{Cov}\left(f(U(t_1\,,x_1),\ldots,U(t_n\,,x_n))\,,g(U(t_1\,,x_1),\ldots,U(t_n\,,x_n))\right) \geq 0, \qquad \text{(FKG)}$$

for all coordinatewise nondecreasing functions $f, g : \mathbb{R}^n \to \mathbb{R}$. (Esary-Proschan-Walkup-1967)

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- Using Clark-Ocone formua and Ito's isometry

$$\begin{aligned}&\mathrm{Cov}\left(f(U(t_1\,,x_1),\ldots,U(t_n\,,x_n))\,,g(U(t_1\,,x_1),\ldots,U(t_n\,,x_n))\right)\\&= \mathrm{E}\left[\int_0^\infty\int_{\mathbb{R}} \mathrm{E}\left[D_{s,y}f(U(t_1\,,x_1),\ldots,U(t_n\,,x_n))|\mathcal{F}_s\right]\mathrm{E}\left[D_{s,y}g(U(t_1\,,x_1),\ldots,U(t_n\,,x_n))|\mathcal{F}_s\right]\mathrm{d}y\mathrm{d}s\right]\end{aligned}$$

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&\mathrm{Cov}\left(f(U(t_1\,,x_1),\ldots,U(t_n\,,x_n))\,,g(U(t_1\,,x_1),\ldots,U(t_n\,,x_n))\right)\\
&= \mathrm{E}\left[\int_0^\infty \int_{\mathbb{R}} \mathrm{E}\left[D_{s,y}f(U(t_1\,,x_1),\ldots,U(t_n\,,x_n))|\mathcal{F}_s\right] \mathrm{E}\left[D_{s,y}g(U(t_1\,,x_1),\ldots,U(t_n\,,x_n))|\mathcal{F}_s\right] \mathrm{d}y\mathrm{d}s\right]\\
&= \sum_{j,\ell=1}^n \mathrm{E}\Bigg[\int_0^\infty \int_{\mathbb{R}} \mathrm{E}\left[\partial_j f(U(t_1\,,x_1),\ldots,U(t_n\,,x_n)) D_{s,y}U(t_j, x_j)|\mathcal{F}_s\right]\\
&\qquad\qquad \mathrm{E}\left[\partial_\ell g(U(t_1\,,x_1),\ldots,U(t_n\,,x_n)) D_{s,y}U(t_\ell, x_\ell)|\mathcal{F}_s\right]\Bigg]\\
&\geq 0.
\end{aligned}$$

- Lebowitz's inequality (1972)

$$\mathrm{P}\{\log U(t\,,x_1) \le a_1, \log U(t\,,x_2) \le a_2, \ldots, \log U(t\,,x_n) \le a_n\} - \prod_{j=1}^{n} \mathrm{P}\{\log U(t\,,x_j) \le a_j\}$$

$$\le \sum_{1\le j<k\le n} \left(\mathrm{P}\{\log U(t\,,x_j) \le a_j, \log U(t\,,x_k) \le a_k\} - \mathrm{P}\{\log U(t\,,x_j) \le a_j\}\mathrm{P}\{\log U(t\,,x_k) \le a_k\}\right)$$

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- $\log U(t\,,0)$ has a bounded and continuous probability density function by Malliavin calculus.
- By an inequality by Bagai-Prakasa Rao-1991 for associated random variables with bounded pdf,

$$\sup_{a,b\in\mathbb{R}} \left(\mathrm{P}\left\{\log U(t\,,x) \leq a, \log U(t\,,y) \leq b\right\} - \mathrm{P}\left\{\log U(t\,,x) \leq a\right\} \mathrm{P}\left\{\log U(t\,,y) \leq b\right\}\right)$$

$$\leq K \left[\mathrm{Cov}(\log U(t\,,x)\,, \log U(t\,,y))\right]^{1/3}.$$

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- By Poincaré inequality,

$$\begin{aligned}\mathrm{Cov}(\log U(t\,,x)\,, \log U(t\,,y)) &\le \int_0^t \int_{\mathbb{R}} \|D_{r,z} \log U(t\,,x)\|_2 \|D_{r,z} \log U(t\,,y)\|_2 \mathrm{d}z\mathrm{d}r \\ &\lesssim \frac{1}{|x-y|}.\end{aligned}$$

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$$\le \sum_{1\le j<k\le n} \left(\mathrm{P}\{\log U(t\,,x_j) \le a_j, \log U(t\,,x_k) \le a_k\} - \mathrm{P}\{\log U(t\,,x_j) \le a_j\}\mathrm{P}\{\log U(t\,,x_k) \le a_k\}\right)$$

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- Remark: $\mathrm{Cov}(U(t\,,x)\,, U(t\,,y)) \asymp \frac{1}{|x-y|}$.

Proof of lower bound

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Let $\beta \in (0, \frac{1}{8}\sqrt{\frac{t}{2}})$. Choose and fix $a \in (0, \frac{1}{6})$ and $\epsilon \in (0,1)$ such that $\beta < \frac{a}{\frac{4}{3}\sqrt{\frac{2}{t}+\epsilon}}$.

Let $x_j = jN/\lfloor N^a \rfloor$ for $j = 1, \ldots, \lfloor N^a \rfloor$.

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$$\mathrm{P}\left\{\max_{0\le x\le N} \log U(t\,,x) \le (\beta \log N)^{2/3}\right\} \le \mathrm{P}\left\{\max_{1\le j\le \lfloor N^a \rfloor} \log U(t\,,x_j) \le (\beta \log N)^{2/3}\right\}$$

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$$= \mathrm{P}\left\{\max_{1\le j\le \lfloor N^a \rfloor} \log U(t\,,x_j) \le (\beta \log N)^{2/3}\right\} - \prod_{j=1}^{\lfloor N^a \rfloor} \mathrm{P}\left\{\log U(t\,,x_j) \le (\beta \log R)^{2/3}\right\}$$

$$+ \left(1 - \mathrm{P}\left\{\log U(t\,,0) > (\beta \log N)^{2/3}\right\}\right)^{\lfloor N^a \rfloor}$$

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&\quad = \mathrm{P}\left\{\max_{1\le j\le \lfloor N^a\rfloor} \log U(t\,,x_j) \le (\beta \log N)^{2/3}\right\} - \prod_{j=1}^{\lfloor N^a\rfloor} \mathrm{P}\left\{\log U(t\,,x_j) \le (\beta \log R)^{2/3}\right\} \\
&\qquad\qquad + \left(1 - \mathrm{P}\left\{\log U(t\,,0) > (\beta \log N)^{2/3}\right\}\right)^{\lfloor N^a\rfloor} \\
&\quad \lesssim \sum_{1\le j<k\le \lfloor N^a\rfloor} [\mathrm{Cov}(\log U(t\,,x_j)\,, \log U(t\,,x_k))]^{1/3} + \left(1 - \mathrm{e}^{-(\frac{4}{3}\sqrt{\frac{2}{t}}+\epsilon)\beta\log N}\right)^{\lfloor N^a\rfloor} \\
&\quad \lesssim N^{2a-\frac{1}{3}} + \mathrm{e}^{-\frac{1}{2}N^{a-\frac{4}{3}(\sqrt{\frac{2}{t}}+\epsilon)\beta}}.
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$$
\begin{aligned}
\mathrm{P}\left\{\max_{0\le x\le N} \log U(t\,,x) \le (\beta \log N)^{2/3}\right\} &\le \mathrm{P}\left\{\max_{1\le j\le \lfloor N^a\rfloor} \log U(t\,,x_j) \le (\beta \log N)^{2/3}\right\} \\
&= \mathrm{P}\left\{\max_{1\le j\le \lfloor N^a\rfloor} \log U(t\,,x_j) \le (\beta \log N)^{2/3}\right\} - \prod_{j=1}^{\lfloor N^a\rfloor} \mathrm{P}\left\{\log U(t\,,x_j) \le (\beta \log R)^{2/3}\right\} \\
&\quad + \left(1 - \mathrm{P}\left\{\log U(t\,,0) > (\beta \log N)^{2/3}\right\}\right)^{\lfloor N^a\rfloor} \\
&\lesssim \sum_{1\le j<k\le \lfloor N^a\rfloor} [\mathrm{Cov}(\log U(t\,,x_j)\,, \log U(t\,,x_k))]^{1/3} + \left(1 - \mathrm{e}^{-(\frac{4}{3}\sqrt{\frac{2}{t}}+\epsilon)\beta \log N}\right)^{\lfloor N^a\rfloor} \\
&\lesssim N^{2a-\frac{1}{3}} + \mathrm{e}^{-\frac{1}{2}N^{a-\frac{4}{3}(\sqrt{\frac{2}{t}}+\epsilon)\beta}}.
\end{aligned}
$$

Remark: by the ergodicity of $\{U(t\,,x) : x \in \mathbb{R}\}$ (Chen-Khoshnevisan-Nualart-P.-2022), there exist finite positive constants α_1, α_2 such that a.s.

$$
\liminf_{N\to\infty} \frac{\max_{0\le x\le N} \log U(t\,,x)}{(\log N)^{2/3}} = \alpha_1, \quad \limsup_{N\to\infty} \frac{\max_{0\le x\le N} \log U(t\,,x)}{(\log N)^{2/3}} = \alpha_2.
$$